Transmit Optimization with Improper Gaussian Signaling in Multiuser Interference Channels

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MIMO Interference Channel (MIMO-IC)

- **$K$-user MIMO-IC.** Applications include, e.g.,
  - Coordinated MIMO downlink transmission by multiple BSs (Multi-cell)
  - Multiuser MIMO downlink linear precoding (Single-cell)

\[
y_k = H_{kk} x_k + \sum_{j \neq k} H_{kj} x_j + n_k, \quad k = 1, \cdots, K.
\]

- $y_k \in \mathbb{C}^{N \times 1}$: received signal vector at RX $k$.
- $x_k \in \mathbb{C}^{M \times 1}$: transmitted signal vector by TX $k$.
- $H_{kk} \in \mathbb{C}^{N \times M}$: direct channel for the $k$th user TX and RX pair.
- $H_{kj} \in \mathbb{C}^{N \times M}$: cross channel from TX $j$ to RX $k$, $k \neq j$.
- $n_k \in \mathbb{C}^{N \times 1}$: additive white Gaussian noise (AWGN) at RX $k$. $n_k \sim \mathcal{CN}(0, \sigma^2 I_N)$.
Information-theoretical

- Capacity region unknown in general
- \textbf{Strong} interference: interference decoding [Carleial75] [Sato81] [ShangPoor12]
- Inner bound: Han-Kobayashi rate splitting [HanKobayashi81], capacity within 1 bit [EtkinTseWang08]
- Degrees of freedom (DoF): interference alignment [CadambeJafar08]
Related Work (1/3)

- **Information-theoretical**
  - Capacity region unknown in general
  - **Strong** interference: interference decoding [Carleial75] [Sato81] [ShangPoor12]
  - Inner bound: Han-Kobayashi rate splitting [HanKobayashi81], capacity within 1 bit [EtkinTseWang08]
  - Degrees of freedom (DoF): interference alignment [CadambeJafar08]

- **Interference treated as noise**
  - Sum capacity optimal for **noisy** interference channel [ShangKramerChen09]
  - Transmit beamforming with SINR constraint: uplink-downlink duality [DahroujYu10]
  - Minimum mean square error (MMSE) [ShenLiTaoWang10]
  - (Weighted) sum-rate maximization (WSRMax)
  - Achievable rate region
(Weighted) sum-rate maximization (WSRMax)

- NP-hard in general [LiuDaiLuo11]
- Optimal binary power allocation for two-user SISO-IC [GjendemsjøGesbertØienKiani08]
- Gradient descent for precoding or transmit covariance matrices for MIMO-IC [YeBlum03][SungLeeParkLee10]
- Iterative weighted-MMSE approach [ShiRazaviyaynLuoHe11] [RazaviyaynSanjabiLuo12]
- Interference pricing [HuangBerryHonig06], virtual SINR framework [ZakhourGesbert09]
- Global optimal via \textit{monotonic optimization} [QianZhangHuang09] [JorswieckLarsson10] [LiuZhangChua12] [BjornsonZhengBengtssonOttersten12]
Achievable rate region

- SISO-IC [CharafeddineSezginPaulraj07]
- MISO-IC
  - More transmit antennas than number of users ($M > K$) [JorswieckLarssonDanev08]
  - WSRMax approach [ShangChenPoor11]
  - Rate-profile approach, interference-temperature approach [ZhangCui10]
  - Optimality of transmit beamforming (rank-1 transmit covariance matrix) [ZhangCui10] [ShangChenPoor11] [MochaourabJorswi11]
- MIMO-IC [BjörnsonBengtssonOttersten12] [CaoJorswieckShi] [ParkSung]
Achievable Rate: Prior Result

- Achievable rate expression with interference treated as noise:

\[
R_k = \log \frac{\sigma^2 I + \sum_{j=1}^{K} H_{kj} C_{xj} H_{kj}^H}{\sigma^2 I + \sum_{j \neq k} H_{kj} C_{xj} H_{kj}^H}, \quad k = 1, \ldots, K
\]

- \(x_k \sim \mathcal{CN}(0, C_{x_k}), \forall k: \) Circularly Symmetric Complex Gaussian (CSCG)

- \(C_{x_k}: \) covariance matrix, i.e., \(C_{x_k} = \mathbb{E}(x_k x_k^H)\)
Achievable Rate: Prior Result

- Achievable rate expression with interference treated as noise:
  \[
  R_k = \log \frac{|\sigma^2 I + \sum_{j=1}^{K} H_{kj} C_{xj} H_{kj}^H|}{|\sigma^2 I + \sum_{j \neq k} H_{kj} C_{xj} H_{kj}^H|}, \ k = 1, \ldots, K
  \]

- \(x_k \sim \mathcal{CN}(0, C_{xk})\), \(\forall k\): Circularly Symmetric Complex Gaussian (CSCG)

- \(C_{xk}\): covariance matrix, i.e., \(C_{xk} = \mathbb{E}(x_k x_k^H)\)

- MISO-IC: \(R_k = \log \left(1 + \frac{|h_{kk} C_{xk} h_{kk}^H|}{\sigma^2 + \sum_{j \neq k} |h_{kj} C_{xj} h_{kj}^H|}\right)\)

  - Beamforming (\(C_{xk} = t_k t_k^H\)): \(R_k = \log \left(1 + \frac{|h_{kk} t_k|^2}{\sigma^2 + \sum_{j \neq k} |h_{kj} t_j|^2}\right)\)

- SISO-IC: \(R_k = \log \left(1 + \frac{|h_{kk}|^2 C_{xk}}{\sigma^2 + \sum_{j \neq k} |h_{kj}|^2 C_{xj}}\right)\). \(C_{xk}\): power of user \(k\)
Achievable Rate: Revisit

$$R_k = \log \frac{\left| \sigma^2 I + \sum_{j=1}^{K} H_{kj} C_{xj} H_{kj}^H \right|}{\left| \sigma^2 I + \sum_{j \neq k} H_{kj} C_{xj} H_{kj}^H \right|}$$ (1)

**Question:** Is (1) the best achievable rate with single-user decoding?
Achievable Rate: Revisit

\[ R_k = \log \left| \frac{\sigma^2 I + \sum_{j=1}^{K} H_{kj} C_{x_j} H_{kj}^H}{\sigma^2 I + \sum_{j \neq k} H_{kj} C_{x_j} H_{kj}^H} \right| \]  

Question: Is (1) the best achievable rate with single-user decoding?

Answer: No

One important assumption of (1): \( x_k \sim \mathcal{CN}(0, C_{x_k}) \), Circularly Symmetric Complex Gaussian (CSCG), or proper Gaussian

Proper and Improper Random Vectors

- Zero-mean complex-valued random vector (RV): \( z = u + jv \)
- Covariance matrix: \( C_z \triangleq \mathbb{E}(zz^H) \)
- Pseudo-covariance matrix: \( \tilde{C}_z \triangleq \mathbb{E}(zz^T) \)
- In general, both \( C_z \) and \( \tilde{C}_z \) are required for complete second-order characterization [1]. Let \( R_{xy} = \mathbb{E}(xy^T) \), then

\[
\begin{align*}
R_{uu} &= \frac{1}{2} \Re \{ C_z + \tilde{C}_z \}, \\
R_{vv} &= \frac{1}{2} \Re \{ C_z - \tilde{C}_z \}, \\
R_{vu} &= \frac{1}{2} \Im \{ C_z + \tilde{C}_z \}, \\
R_{uv} &= -\frac{1}{2} \Im \{ C_z - \tilde{C}_z \}.
\end{align*}
\]
Proper and Improper Random Vectors

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- Covariance matrix: \( C_z \triangleq \mathbb{E}(zz^H) \)
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- In general, both \( C_z \) and \( \tilde{C}_z \) are required for complete second-order characterization [1]. Let \( R_{xy} = \mathbb{E}(xy^T) \), then

\[
R_{uu} = \frac{1}{2} \Re \{ C_z + \tilde{C}_z \}, \quad R_{vv} = \frac{1}{2} \Re \{ C_z - \tilde{C}_z \}, \\
R_{vu} = \frac{1}{2} \Im \{ C_z + \tilde{C}_z \}, \quad R_{uv} = -\frac{1}{2} \Im \{ C_z - \tilde{C}_z \}.
\]

Definition

[1]: **Proper**: A complex RV \( z \) is called proper if \( \tilde{C}_z = 0 \); otherwise, it is called improper.

- Proper \( \iff \) \( R_{uu} = R_{vv}, \ R_{uv} = -R_{uv}^T \)
Proper Versus Circularly Symmetric

Definition

[2]: **Circularly Symmetric**: A complex RV \( z \) is circularly symmetric if \( z \) and \( z' = e^{j\alpha}z \) have the same distribution for any real value \( \alpha \).
Proper Versus Circularly Symmetric

Definition

[2]: **Circularly Symmetric**: A complex RV \( z \) is circularly symmetric if \( z \) and \( z' = e^{j\alpha}z \) have the same distribution for any real value \( \alpha \).

- For circularly symmetric RV \( z \), we have
  \[
  \tilde{C}_z = \tilde{C}_{z'} = \mathbb{E}(z'z'^T) = e^{j2\alpha} \tilde{C}_z, \quad \forall \alpha, \implies \tilde{C}_z = 0.
  \]

- So circularity \( \implies \) properness
- But properness \( \nRightarrow \) circularity
- **For Zero-Mean Gaussian**: Circularity \( \iff \) Properness
- \( x_k \sim \mathcal{CN}(0, C_{x_k}) \) (CSCG) \( \iff \) \( x_k \) zero-mean proper Gaussian (\( \tilde{C}_{x_k} = 0 \))
For an arbitrary complex RV $z$, define the augmented covariance matrix

$$ C_{z} \triangleq \mathbb{E}\left(\left[\begin{array}{c} z \\
 z^{*} \end{array}\right]\left[\begin{array}{c} z \\
 z^{*} \end{array}\right]^H\right) = \left[\begin{array}{cc} C_{z} & \tilde{C}_{z} \\
 \tilde{C}_{z}^{*} & C_{z}^{*} \end{array}\right] $$

The entropy of complex Gaussian RV $z \in \mathbb{C}^n$ is [2]

$$ h(z) = \frac{1}{2} \log\left((\pi e)^{2n} |C_{z}|\right) $$

For proper Gaussian RV ($\tilde{C}_{z} = 0$), entropy reduces to

$$ h(z) = \log\left((\pi e)^{n} |C_{z}|\right) $$
Why Proper Gaussian Signaling?

- Why is proper complex Gaussian (or CSCG) signaling usually assumed in the literature?
  - Noise is modeled as proper Gaussian, \( \mathbf{n} \sim \mathcal{CN}(\mathbf{0}, \mathbf{C}_n) \)
  - Maximum-entropy theorem [1], \( \mathbf{z} \in \mathbb{C}^n \), given \( \mathbb{E}(\mathbf{zz}^H) = \mathbf{C}_z \),
    \[
    h(\mathbf{z}) \leq \log [(\pi e)^n|\mathbf{C}_z|], \quad (= \text{iff } \mathbf{z} \sim \mathcal{CN}(\mathbf{0}, \mathbf{C}_z))
    \]
For ICs, **improper** complex Gaussian signaling achieves
- higher DoF for 3-user time-invariant SISO-IC [6]
- higher rate for 2-user SISO-IC [7, 8]

Existing approach:

complex-valued SISO-IC $\rightarrow$ real-valued MIMO-IC

Our approach: complex-valued covariance and pseudo-covariance matrices optimization
- More insights
- Guaranteed performance gain over proper Gaussian signaling
Achievable Rate: Improper Gaussian Signaling

\[ y_k = H_{kk}x_k + \sum_{j \neq k} H_{kj}x_j + n_k, \quad \mathbb{E}(x_k x_k^H) = C_{x_k}, \quad \mathbb{E}(x_k x_k^T) = \widetilde{C}_{x_k}, \quad \forall k. \]
Achievable Rate: Improper Gaussian Signaling

\[ y_k = H_{kk}x_k + \sum_{j \neq k} H_{kj}x_j + n_k, \quad \mathbb{E}(x_kx_k^H) = C_{x_k}, \quad \mathbb{E}(x_kx_k^T) = \tilde{C}_{x_k}, \quad \forall k. \]

\[ R_k = \log \frac{\sigma^2 I + \sum_{j=1}^{K} H_{kj}C_{x_j}H_{kj}^H}{\sigma^2 I + \sum_{j \neq k} H_{kj}C_{x_j}H_{kj}^H} + \frac{1}{2} \log \frac{|I - C_{y_k}^{-1}\tilde{C}_{y_k}C_{y_k}^{-T}\tilde{C}_{y_k}^H|}{|I - C_{s_k}^{-1}\tilde{C}_{s_k}C_{s_k}^{-T}\tilde{C}_{s_k}^H|} \]

\[ \triangleq R_{k,\text{proper}}(\{C_{x_j}\}) \]

\[ C_{y_k} = \mathbb{E}(y_ky_k^H) = \sum_{j=1}^{K} H_{kj}C_{x_j}H_{kj}^H + \sigma^2 I, \quad \tilde{C}_{y_k} = \mathbb{E}(y_ky_k^T) = \sum_{j=1}^{K} H_{kj}\tilde{C}_{x_j}H_{kj}^T \]

\[ C_{s_k} = \mathbb{E}(s_ks_k^H) = \sum_{j \neq k} H_{kj}C_{x_j}H_{kj}^H + \sigma^2 I, \quad \tilde{C}_{s_k} = \mathbb{E}(s_ks_k^T) = \sum_{j \neq k} H_{kj}\tilde{C}_{x_j}H_{kj}^T \]
Achievable Rate: Improper Gaussian Signaling

\[ R_k = \log \left( \frac{\sigma^2 I + \sum_{j=1}^{K} H_{kj} C_{xj} H_{kj}^H}{\sigma^2 I + \sum_{j \neq k} H_{kj} C_{xj} H_{kj}^H} \right) + \frac{1}{2} \log \left( \frac{|I - C_{yk}^{-1} \tilde{C}_{yk} C_{yk}^{-T} \tilde{C}_{yk}^H|}{|I - C_{sk}^{-1} \tilde{C}_{sk} C_{sk}^{-T} \tilde{C}_{sk}^H|} \right) \]

- Additional term (in red) not present in proper Gaussian signaling
- Special case: \( \tilde{C}_{xk} = 0, \forall k \), i.e., proper Gaussian signaling
- Rate improvement over proper Gaussian signaling by choosing \( \{\tilde{C}_{xj}\} \) to make the additional term positive
Achievable Rate Region: Improper Gaussian Signaling

- $C_{x_k}$ and $\tilde{C}_{x_k}$ is a valid pair of covariance and pseudo-covariance matrices if and only if (iff) [2]

$$C_{x_k} \triangleq \begin{bmatrix} C_{x_k} & \tilde{C}_{x_k} \\ \tilde{C}_{x_k}^* & C_{x_k}^* \end{bmatrix} \succeq 0$$

- Conditions that $C_{x_k} \succeq 0$ and $\tilde{C}_{x_k}$ symmetric are implied by $C_{x_k} \succeq 0$

- Power constraint: $\text{Tr}\{C_{x_k}\} \leq P_k, \forall k$

- Achievable rate region:

$$\mathcal{R} \triangleq \bigcup_{\text{Tr}\{C_{x_k}\} \leq P_k, \atop C_{x_k} \succeq 0, \forall k} \left\{ (r_1, \ldots, r_K) : 0 \leq r_k \leq R_k, \forall k \right\}$$
Problem Formulation: Rate-Profile Approach to Characterize Rate-Region Pareto Boundary

- Given a rate profile $\alpha = (\alpha_1 \cdots \alpha_K) \succeq 0$, $\sum_{k=1}^{K} \alpha_k = 1$, the corresponding Pareto-optimal rate tuple can be found by solving

$$\begin{align*}
\max R \\
\{C_{x_k}, \tilde{C}_{x_k}, \bar{C}_{x_k}\}, R
\end{align*}$$

s.t. $R_k \geq \alpha_k R, \ \forall k,$

$$\text{Tr}(C_{x_k}) \leq P_k, \ \forall k,$$

$$\begin{bmatrix}
C_{x_k} & \tilde{C}_{x_k} \\
\tilde{C}_{x_k}^* & C_{x_k}^*
\end{bmatrix} \succeq 0, \ \forall k,$$

- $R^* \cdot \alpha$ is on Pareto boundary of rate region [9]
- Circumvent non-convexity of traditional weighted sum-rate maximization (WSRMax)
Conventional proper Gaussian signaling: linear precoding with optimized covariance matrix $C_{x_k}$

\[ d_k \sim \mathcal{CN}(0, I) \]

\[ C_{x_k}^{\frac{1}{2}} \]

\[ x_k \sim \mathcal{CN}(0, C_{x_k}) \]

Transmitted Signal Vector

\[ C_{x_k} = UDU^H \implies C_{x_k}^{\frac{1}{2}} = UD^{\frac{1}{2}} \]
Widely Linear Precoding

- **Conventional proper** Gaussian signaling: linear precoding with optimized covariance matrix $\mathbf{C}_{x_k}$

$$d_k \sim \mathcal{CN}(0, \mathbf{I}) \quad \xrightarrow{\text{Information-Bearing Symbols}} \quad \mathbf{C}_{x_k}^{\frac{1}{2}} \quad \xrightarrow{\text{Transmitted Signal Vector}} \quad x_k \sim \mathcal{CN}(0, \mathbf{C}_{x_k})$$

$$\mathbf{C}_{x_k} = \mathbf{C}_{x_k}^{\frac{1}{2}} (\mathbf{C}_{x_k}^{\frac{1}{2}})^H$$, via Eigenvalue Decomposition (EVD):

$$\mathbf{C}_{x_k} = \mathbf{U} \mathbf{D} \mathbf{U}^H \quad \Rightarrow \quad \mathbf{C}_{x_k}^{\frac{1}{2}} = \mathbf{U} \mathbf{D}^{\frac{1}{2}}$$

- **Improper** Gaussian signaling: with optimized $\mathbf{C}_{x_k}$ and $\tilde{\mathbf{C}}_{x_k}$, how to obtain $\mathbf{z}_k$ such that

$$d_k \sim \mathcal{CN}(0, \mathbf{I}) \quad \xrightarrow{\text{Information-Bearing Symbols}} \quad ?? \quad \xrightarrow{\text{Transmitted Signal Vector}} \quad \mathbf{z}_k : \mathbb{E}(\mathbf{z}_k \mathbf{z}_k^H) = \mathbf{C}_{x_k}$$

$$\mathbb{E}(\mathbf{z}_k \mathbf{z}_k^T) = \tilde{\mathbf{C}}_{x_k}$$
Widely Linear Precoding

- Linear precoding:
  \[ z_k = U_k d_k \implies \mathbb{E}(z_k z_k^H) = U_k \tilde{C}_d U_k^T = 0 \]

- So conventional linear precoding is *not* sufficient

- With augmented covariance matrix, we need to have
  \[ C_{z_k} = \mathbb{E}\left( \begin{bmatrix} z_k & z_k^* \end{bmatrix} \begin{bmatrix} z_k & z_k^* \end{bmatrix}^H \right) = \begin{bmatrix} C_{x_k} & \tilde{C}_{x_k} \\ \tilde{C}_{x_k}^* & C_{x_k}^* \end{bmatrix} = C_{x_k} \] (2)

- (2) is satisfied with mapping
  \[ \begin{bmatrix} z_k \\ z_k^* \end{bmatrix} = C_{x_k}^{1/2} \begin{bmatrix} d_k \\ d_k^* \end{bmatrix} \]
Widely Linear Precoding

\[
\begin{bmatrix}
  z_k \\
  z_k^*
\end{bmatrix} = \underbrace{C_{x_k}^{\frac{1}{2}}} \begin{bmatrix}
  d_k \\
  d_k^*
\end{bmatrix}
\] (3)

- Finding $C_{x_k}^{\frac{1}{2}}$ via conventional EVD cannot satisfy (3) in general due to the complex-conjugate relation between the top and bottom blocks.

- Instead, need to find $C_{x_k}^{\frac{1}{2}}$ with the following structure:

$$C_{x_k}^{\frac{1}{2}} = \begin{bmatrix}
  B_1 & B_2 \\
  B_2^* & B_1^*
\end{bmatrix}$$

- So that (3) can be satisfied since it is equivalent to the following two consistent equations:

\[
\begin{align*}
  z_k &= B_1 d_k + B_2 d_k^*, \\
  z_k^* &= B_2^* d_k + B_1^* d_k^*
\end{align*}
\]
Widely Linear Precoding

Theorem

[10] There exists one form of EVD for the augmented covariance matrix $\mathbf{C}_{x_k} \in \mathbb{C}^{2M \times 2M}$ such that

$$
\mathbf{C}_{x_k} = (TV)\Lambda(TV)^H,
$$

where $T \triangleq \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{I}_M & i\mathbf{I}_M \\ \mathbf{I}_M & -i\mathbf{I}_M \end{bmatrix}$, $V \in \mathbb{R}^{2M \times 2M}$: real-valued orthogonal matrix, $
\Lambda$: eigenvalues.

Then we have $\mathbf{C}_{x_k}^{1/2} = T(V\Lambda^{1/2})T^H = \begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 \\ \mathbf{B}_2^* & \mathbf{B}_1^* \end{bmatrix}$

- The specific structure of $\mathbf{C}_{x_k}^{1/2}$ is satisfied due to the special unitary matrix $T$, and the fact that $V\Lambda^{1/2}$ is real-valued.
Widely Linear Precoding

\[
\begin{bmatrix}
  z_k \\
  z_k^*
\end{bmatrix} = C_{x_k}^{1/2} \begin{bmatrix}
  d_k \\
  d_k^*
\end{bmatrix}
\]

- \( C_{x_k}^{1/2} = T(V^{1/2}) T^H = \begin{bmatrix}
  B_1 & B_2 \\
  B_2^* & B_1^*
\end{bmatrix} \)

- Widely linear precoding: \( z_k = B_1 d_k + B_2 d_k^* \)

\( d_k \sim \mathcal{CN}(0, I) \)

\( z_k : \mathbb{E}(z_k z_k^H) = C_{x_k} \)

\( \mathbb{E}(z_k z_k^T) = \tilde{C}_{x_k} \)
Case Study I: $K$-User SISO-IC

- $K$-user SISO-IC:

$$y_k = h_{kk} x_k + \sum_{j \neq k} h_{kj} x_j + n_k$$

$$C_{x_k} = \mathbb{E}(x_k x_k^*), \quad \tilde{C}_{x_k} = \mathbb{E}(x_k x_k)$$

- Rate expression reduces to

$$R_k = \frac{1}{2} \log \left( \frac{C_{y_k}^2 - |\tilde{C}_{y_k}|^2}{C_{s_k}^2 - |\tilde{C}_{s_k}|^2} \right)$$

$$= \log \left( 1 + \frac{|h_{kk}|^2 C_{x_k}}{\sigma^2 + \sum_{j \neq k} |h_{kj}|^2 C_{x_j}} \right) + \frac{1}{2} \log \frac{1 - C_{y_k}^{-2} |\tilde{C}_{y_k}|^2}{1 - C_{s_k}^{-2} |\tilde{C}_{s_k}|^2}.$$
The Pareto boundary problem for SISO-IC reduces to

\[
\begin{align*}
\text{max} \quad & R \\
\{C_{x_k}\}, \{\tilde{C}_{x_k}\}, & R \\
s.t. \quad & R_k \geq \alpha_k R, \quad \forall k \\
& 0 \leq C_{x_k} \leq P_k, \quad \forall k \\
& |\tilde{C}_{x_k}|^2 \leq C_{x_k}^2, \quad \forall k
\end{align*}
\]

\[
R_k = \frac{1}{2} \log \frac{C_{y_k}^2 - |\tilde{C}_{y_k}|^2}{C_{s_k}^2 - |\tilde{C}_{s_k}|^2}
\]
The Pareto boundary problem for SISO-IC reduces to

\[
\begin{align*}
\max_{\{C_{x_k}\},\{\tilde{C}_{x_k}\}, R} & \quad R \\
\text{s.t.} & \quad R_k \geq \alpha_k R, \quad \forall k \\
& \quad 0 \leq C_{x_k} \leq P_k, \quad \forall k \\
& \quad |\tilde{C}_{x_k}|^2 \leq C_{x_k}^2, \quad \forall k
\end{align*}
\]

\[
R_k = \frac{1}{2} \log \frac{C_{y_k}^2 - |\tilde{C}_{y_k}|^2}{C_{s_k}^2 - |\tilde{C}_{s_k}|^2}
\]

Approximate solution by semidefinite relaxation (SDR)

\[
\begin{align*}
\max_{\{C_{x_k}\},\{\tilde{C}_{x_k}\}} & \quad \min_{k=1,\ldots,K} \quad \frac{1}{2} \alpha_k \log \frac{C_{y_k}^2 - |\tilde{C}_{y_k}|^2}{C_{s_k}^2 - |\tilde{C}_{s_k}|^2} \\
\text{s.t.} & \quad 0 \leq C_{x_k} \leq P_k, \quad \forall k \\
& \quad |\tilde{C}_{x_k}|^2 \leq C_{x_k}^2, \quad \forall k.
\end{align*}
\]
SDR: the Main Concept

Consider a general quadratically constrained quadratic program (QCQP)

$$\min_{x \in \mathbb{C}^n} x^H C x$$

s.t. $x^H A_i x \leq b_i$, $i = 1, \cdots, m$,

where $C, A_i \in \mathbb{H}^n$, but not necessarily positive semidefinite.
SDR: the Main Concept

Consider a general quadratically constrained quadratic program (QCQP)

\[
\begin{align*}
\min_{x \in \mathbb{C}^n} & \quad x^H C x \\
\text{s.t.} & \quad x^H A_i x \leq b_i, \quad i = 1, \ldots, m,
\end{align*}
\]

where \( C, A_i \in \mathbb{H}^n \), but not necessarily positive semidefinite.

With the identity \( x^H C x = \text{Tr}(Cxx^H) \), \( x^H A_i x = \text{Tr}(A_ixx^H) \), the QCQP problem can be written as [11]

\[
\begin{align*}
\min_{X \in \mathbb{H}^n} & \quad \text{Tr}(CX) \\
\text{s.t.} & \quad X \succeq 0, \quad \text{Tr}(A_iX) \leq b_i, \quad i = 1, \ldots, m, \\
& \quad \text{rank}(X) = 1.
\end{align*}
\]
Drop the rank-one constraint to obtain a relaxed problem

\[
\text{(SDR): } \min_{X \in \mathbb{H}^n} \operatorname{Tr}(CX)
\]
\[
\text{s.t. } X \succeq 0, \quad \operatorname{Tr}(A_i X) \leq b_i, \quad i = 1, \ldots, m,
\]

which is convex.
SDR: the Main Concept

- Drop the rank-one constraint to obtain a relaxed problem

\[
(SDR): \min_{X \in \mathbb{H}^n} \quad \text{Tr}(CX) \\
\text{s.t. } X \succeq 0, \quad \text{Tr}(A_i X) \leq b_i, \quad i = 1, \cdots, m,
\]

which is convex.

- If an SDR solution $X^*$ is of rank one, i.e., $X^* = xx^H$, then $x^*$ is the solution to the original QCQP.

- Otherwise, an approximate solution $\hat{x}$ to QCQP can be generated from $X^*$ with Gaussian randomization [11].
Joint Covariance and Pseudo-Covariance Optimization for SISO-IC

- With some manipulations, the problem for SISO-IC can be formulated as

\[
\max_{c,q,t} \min_k \frac{1}{2\alpha_k} \log \left( \frac{(\sigma^2 t + a_k^T c)^2 - q^H F_k q}{(\sigma^2 t + b_k^T c)^2 - q^H G_k q} \right)
\]

s.t. \( c^T E_k c \leq P_k^2 \), \( e_k^T ct \geq 0 \), \( \forall k \)

\( q^H E_k q \leq c^T E_k c \), \( \forall k \), \( t^2 = 1 \).

\( c \in \mathbb{R}^K \): covariances, \( q \in \mathbb{C}^K \): pseudo-covariances, \( t \): slack variable
With some manipulations, the problem for SISO-IC can be formulated as

\[
\begin{align*}
\max_{c,q,t} & \quad \min_k \frac{1}{2\alpha_k} \log \left( \frac{(\sigma^2 t + a_k^T c)^2 - q^H F_k q}{(\sigma^2 t + b_k^T c)^2 - q^H G_k q} \right) \\
\text{s.t.} & \quad c^T E_k c \leq P_k^2, \quad e_k^T ct \geq 0, \quad \forall k \\
& \quad q^H E_k q \leq c^T E_k c, \quad \forall k, \quad t^2 = 1.
\end{align*}
\]

- \( c \in \mathbb{R}^K \): covariances, \( q \in \mathbb{C}^K \): pseudo-covariances, \( t \): slack variable
- The SDR problem:

\[
\begin{align*}
\max_{c \in S^{K+1}, Q \in \mathbb{H}^K} & \quad \min_k \frac{1}{2\alpha_k} \log \frac{\text{Tr}(A_k C) - \text{Tr}(F_k Q)}{\text{Tr}(B_k C) - \text{Tr}(G_k Q)} \\
\text{s.t.} & \quad \text{Tr}(\hat{E}_k C) \leq P_k^2, \quad \text{Tr}(K_k C) \geq 0, \quad \forall k \\
& \quad \text{Tr}(E_k Q) \leq \text{Tr}(\hat{E}_k C), \quad \forall k, \quad C_{11} = 1 \\
& \quad C \succeq 0, \quad Q \succeq 0.
\end{align*}
\]
The SDR problem is a quasi-convex, and hence can be optimally solved with bisection method.

An approximate solution to the original covariance and pseudo-covariance optimization problem for the SISO-IC is then obtained with Gaussian randomization [11].
Achievable rate region for two-user SISO-IC, SNR = 10 dB.
Numerical Results

Average max-min rates for two-user SISO-IC.
Case Study II: $K$-User MISO-IC

- $K$-user MISO-IC:

$$y_k = h_{kk}x_k + \sum_{j \neq k} h_{kj}x_j + n_k$$

$$\mathbb{E}(x_kx_k^H) = C_{x_k}, \quad \mathbb{E}(x_kx_k^T) = \tilde{C}_{x_k}$$

- Rate expression reduces to

$$R_k = \log \left( 1 + \frac{h_{kk}C_{x_k}h_{kk}^H}{\sigma^2 + \sum_{j \neq k} h_{kj}C_{x_j}h_{kj}^H} \right) + \frac{1}{2} \log \frac{1 - C_{y_k}^{-2} |\tilde{C}_{y_k}|^2}{1 - C_{s_k}^{-2} |\tilde{C}_{s_k}|^2}$$

$\triangleq R_k^{\text{proper}}(\{C_{x_j}\})$

The Pareto boundary problem for MISO-IC reduces to

\[
\max_{\{\mathbf{C}_{x_k}, \tilde{\mathbf{C}}_{x_k}\}, R} \quad R_k \\
\text{s.t.} \quad R_k \geq \alpha_k R, \quad \forall k \\
\text{Tr}\{\mathbf{C}_{x_k}\} \leq P_k, \quad \forall k \\
\begin{bmatrix} \mathbf{C}_{x_k} & \tilde{\mathbf{C}}_{x_k} \\ \tilde{\mathbf{C}}_{x_k}^* & \mathbf{C}_{x_k}^* \end{bmatrix} \succeq 0, \quad \forall k \\
\begin{bmatrix} h_{kk} \mathbf{C}_{x_k} \mathbf{h}_k^H \\ \sigma^2 + \sum_{j \neq k} h_{kj} \mathbf{C}_{x_j} \mathbf{h}_j^H \end{bmatrix} \quad \Delta \quad R_k^{\text{proper}}(\{\mathbf{C}_{x_j}\}) \\
+ \frac{1}{2} \log \left( 1 - \frac{1}{1 - C_{y_k}^{-2} |\tilde{C}_{y_k}|^2} \right) \\
+ \frac{1}{2} \log \left( 1 - \frac{1}{1 - C_{s_k}^{-2} |\tilde{C}_{s_k}|^2} \right).
\]
Pareto Boundary of MISO-IC with Improper Gaussian Signaling

The Pareto boundary problem for MISO-IC reduces to

$$\max_{\{\mathbf{C}_{x_k}\},\{\tilde{\mathbf{C}}_{x_k}\},\mathbf{R}} R$$

s.t. $R_k \geq \alpha_k R$, $\forall k$

$$\text{Tr}\{\mathbf{C}_{x_k}\} \leq P_k$, $\forall k$

$$\begin{bmatrix} \mathbf{C}_{x_k} & \tilde{\mathbf{C}}_{x_k} \\ \tilde{\mathbf{C}}^*_{x_k} & \mathbf{C}^*_{x_k} \end{bmatrix} \succeq 0$, $\forall k$

Separate covariance and pseudo-covariance optimization

1. Covariance optimization: with $\tilde{\mathbf{C}}_{x_k} = \mathbf{0}$, $\forall k$, optimize covariance matrices $\{\mathbf{C}^*_{x_k}\}$ with proper Gaussian signaling.

2. Pseudo-covariance optimization: fixing $\{\mathbf{C}^*_{x_k}\}$, optimize pseudo-covariance matrices $\{\tilde{\mathbf{C}}_{x_k}\}$ to improve rate over (optimal) proper Gaussian signaling.

$$R_k = \log \left(1 + \frac{h_{kk}\mathbf{C}_{x_k}h^H_{kk}}{\sigma^2 + \sum_{j \neq k} h_{kj}\mathbf{C}_{x_j}h^H_{kj}}\right)$$

$$\triangleq R^\text{proper}_k(\{\mathbf{C}_{x_j}\}) + \frac{1}{2} \log \left(1 - C^{-2}_{y_k} |\tilde{\mathbf{C}}_{y_k}|^2 \right) \left(1 - C^{-2}_{s_k} |\tilde{\mathbf{C}}_{s_k}|^2 \right).$$
Covariance Optimization

With $\tilde{C}_{x_k} = 0$, $\forall k$, the problem reduces to

$$\max_{r, \{C_{x_k}\}} r$$

s.t. $\log \left(1 + \frac{h_{kk} C_{x_k} h^H_{kk}}{\sigma^2 + \sum_{j \neq k} h_{kj} C_{x_j} h^H_{kj}}\right) \geq \alpha_k r, \forall k,$

$$\text{Tr}\{C_{x_k}\} \leq P_k, C_{x_k} \succeq 0, \forall k.$$
Covariance Optimization

- With $\tilde{C}_{x_k} = 0$, $\forall k$, the problem reduces to

$$\begin{align*}
\max_{r,\{C_{x_k}\}} & \quad r \\
\text{s.t.} & \quad \log \left( 1 + \frac{h_{kk} C_{x_k} h_{kk}^H}{\sigma^2 + \sum_{j \neq k} h_{kj} C_{x_j} h_{kj}^H} \right) \geq \alpha_k r, \quad \forall k, \\
\quad & \quad \text{Tr}\{C_{x_k}\} \leq P_k, \quad C_{x_k} \succeq 0, \quad \forall k.
\end{align*}$$

- Since beamforming is optimal, the problem can be solved via the following SOCP feasibility problem together with bisection:

$$\begin{align*}
\text{Find } \{t_k\} \\
\text{s.t.} & \quad \sigma^2 + \sum_{j=1}^{K} |h_{kj} t_j|^2 \leq \left( 1 + \frac{1}{e^{\alpha_k r} - 1} \right) (h_{kk} t_k)^2, \quad \forall k, \\
\quad & \quad \Im\{h_{kk} t_k\} = 0, \quad \|t_k\|^2 \leq P_k, \quad \forall k.
\end{align*}$$
With \( \{r^*, C_{x_k}^* = t_k t_k^H\} \) the solution to the covariance optimization problem, the pseudo-covariance optimization problem is

\[
\begin{aligned}
\max_{R, \{\widetilde{C}_{x_k}\}} & \quad R \\
\text{s.t.} & \quad \alpha_k r^* + \frac{1}{2} \log \frac{1 - C_{y_k}^{-2} |\widetilde{C}_{y_k}|^2}{1 - C_{s_k}^{-2} |\widetilde{C}_{s_k}|^2} \geq \alpha_k R, \quad \forall k \\
& \quad \begin{bmatrix} t_k t_k^H & \widetilde{C}_{x_k} \\ \widetilde{C}_{x_k}^* & (t_k t_k^H)^* \end{bmatrix} \succeq 0, \quad \forall k
\end{aligned}
\]
Pseudo-Covariance Optimization

- With \( \{r^*, C^*_x_k = t_k t_k^H\} \) the solution to the covariance optimization problem, the pseudo-covariance optimization problem is

\[
\begin{align*}
\max_{R, \{\tilde{C}_{x_k}\}} & \quad R \\
\text{s.t.} & \quad \alpha_k r^* + \frac{1}{2} \log \frac{1 - C_{y_k}^{-2}|\tilde{C}_{y_k}|^2}{1 - C_{s_k}^{-2}|\tilde{C}_{s_k}|^2} \geq \alpha_k R, \quad \forall k \\
\begin{bmatrix} t_k t_k^H & \tilde{C}_{x_k} \\ \tilde{C}_{x_k}^* & (t_k t_k^H)^* \end{bmatrix} & \succeq 0, \quad \forall k
\end{align*}
\]

**Lemma**

The positive semidefinite constraint in the above problem is satisfied if and only if

\[
\tilde{C}_{x_k} = Z_k \tilde{t}_k \tilde{t}_k^T, \quad k = 1, \ldots, K,
\]

where \( Z_k \) is a complex scalar variable with constraint \( |Z_k| \leq \|t_k\|^2 \), and \( \tilde{t}_k = t_k / \|t_k\| \) is the normalized (proper) beamforming vector.
Pseudo-Covariance Optimization

\[ \tilde{C}_{x_k} = Z_k \tilde{t}_k \tilde{t}_k^T, \quad k = 1, \ldots, K \]

- Rank-1 pseudo-covariance matrices are optimal.
- \( \tilde{C}_{x_k} \) parameterized by a single scalar variable \( Z_k \) only.
- Number of variables reduced: \( KM^2 \rightarrow K \).
Pseudo-Covariance Optimization

\[ \tilde{C}_{x_k} = Z_k \tilde{t}_k \tilde{t}_k^T, \; k = 1, \ldots, K \]

- Rank-1 pseudo-covariance matrices are optimal.
- \( \tilde{C}_{x_k} \) parameterized by a single scalar variable \( Z_k \) only.
- Number of variables reduced: \( KM^2 \rightarrow K \).
- The problem can be reformulated as

\[
\begin{align*}
\max_{z \in \mathbb{C}^K} \min_{k=1, \ldots, K} & \quad \frac{1}{2} \log \frac{1 - z^H M_k z}{1 - z^H W_k z} \\
\text{s.t.} & \quad |e_k^H z|^2 \leq \| t_k \|^4, \; \forall k,
\end{align*}
\]

where \( z \triangleq [Z_1 \quad \cdots \quad Z_K]^T \).
The SDR of the pseudo-covariance optimization problem:

\[
\text{(SDR)}: \max_{\mathbf{Z} \succeq 0} \min_{k=1,\ldots,K} \frac{1}{2\alpha_k} \log \frac{1 - \text{Tr}(\mathbf{M}_k \mathbf{Z})}{1 - \text{Tr}(\mathbf{W}_k \mathbf{Z})} \quad \text{s.t.} \quad \text{Tr}(\mathbf{E}_k \mathbf{Z}) \leq \| \mathbf{t}_k \|^4, \quad \forall k.
\]

The SDR problem is quasi-convex, which can be solved with bisection.

An approximate solution to the original problem is obtained with Gaussian randomization.

**Lemma**

[12] For two-user MISO-IC ($K = 2$), SDR yields the optimal pseudo-covariance matrices.
Numerical Results

Achievable rate region for two-user MISO-IC, $M = 2$, SNR = 10 dB.
Average max-min rate for three-user MISO-IC, $M = 2$. 
Conclusion and Future Work

Conclusion

- A new rate expression for $K$-user MIMO-IC applicable to the more general improper complex Gaussian signaling
- Widely linear precoding to satisfy the optimized covariance and pseudo-covariance matrices
- Joint covariance and pseudo-covariance optimization for SISO-IC
- Separate covariance and pseudo-covariance optimization for MISO-IC
- Rank-1 optimality of the pseudo-covariance matrices for MISO-IC
- Guaranteed rate gain over proper complex Gaussian signaling

Future work

- More general $K$-user MIMO-IC
- Joint covariance and pseudo-covariance optimization combined with time/frequency symbol extension
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- A new rate expression for $K$-user MIMO-IC applicable to the more general improper complex Gaussian signaling
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