GENERALIZED CONSTRAINT FOR A LOSSLESS MULTIPORT NETWORK

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Abstract—This paper presents the generalized constraint conditions for an arbitrary lossless multiport network in partitioned matrix forms, which involve the constraints at magnitudes, phases and determinant of the scattering matrix. The results show that some special properties of a lossless two-port network, such as quasi-reciprocity, quasi-symmetry, and π-convergence of characteristic phases, can be generalized to a lossless N-port network in partitioned matrix forms. Some examples of lossless networks are given to illustrate the application and validity of the proposed approach in this paper.

1 Introduction

2 Square Submatrix Constraints for a Lossless N-Port Network

3 Non-Square Submatrix Constraints for a Lossless N-Port Network

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1. INTRODUCTION

The investigation of lossless networks is very important in many practical problems and has got a considerable amount of attention for a long time [1–6]. In microwave engineering, many passive and small lossy components can be analyzed as the lossless networks, which
also develop the theories of filters, impedance matching circuits and couplers, etc.

It is well known that the scattering matrix of a lossless network meets the unitary condition that also represents an energy relation

$$[S]^+ [S] = [I]$$  \hspace{1cm} (1)

where symbol + represents conjugation and transposition. For a lossless two-port network, the following constraint conditions may result from (1)

$$[S_{11}] = [S_{22}]$$  \hspace{1cm} (2)
$$[S_{12}] = [S_{21}]$$  \hspace{1cm} (3)

It can be referred to (2) and (3) as quasi-symmetry and quasi-reciprocity, respectively. Let $S_{ij} = |S_{ij}| e^{j\varphi_{ij}}$, where $\varphi_{ij}$ is the phase of the matrix element $S_{ij}$. According to (1) and introducing a characteristic phase, thus we can easily get

$$\Phi_{12} = (\varphi_{12} + \varphi_{21}) - (\varphi_{11} + \varphi_{22}) = \pi$$  \hspace{1cm} (4)

where $\Phi_{12}$ is referred to as the characteristic phase. In general, it can be defined $\Phi_{ij} = (\varphi_{ij} + \varphi_{ji}) - (\varphi_{ii} + \varphi_{jj})$ as the characteristic phase of $i$-$j$ port for a general multiport network. Liang and Guan [7] has proved that the characteristic phases are the intrinsic parameters of a network, which are independent on the reference plane of the wave network ports. Liang [4] first found that the generalized magnitude relation for a lossless and reciprocal $N$-port network in terms of the partitioned matrix forms can be expressed as $|\det(S_{II})| = |\det(S_{III})|$, where $\det(S_{II})$ and $\det(S_{III})$ represent the determinants of submatrix $S_{II}$ and $S_{III}$, respectively. This result has been extended to a lossless and nonreciprocal network by Lin [5]. It is pointed out in [5] that $|\det(S_{II})| = |\det(S_{III})|$ when both $S_{II}$ and $S_{III}$ are also square submatrixes. Subsequently, Shi and Liang [6] derived the generalized phase relation $\det(S) = \exp[j(\varphi_{II} + \varphi_{III})]$, where $\varphi_{II}$ and $\varphi_{III}$ represent the phases of determinant of submatrix $S_{II}$ and $S_{III}$, respectively. However, the constraints for characteristic phase and the generalized magnitude and phase relations of a non-square submatrix have not been discussed. In this paper, the generalized constraint conditions for an arbitrary lossless $N$-port network, involving the characteristic phase and the generalized magnitude and phase relations of a non-square submatrix, are presented in terms of the partitioned matrix forms. To be more important and meaningful, it is found that by using generalized constraint conditions, an analysis of the properties of a lossless $N$-port network becomes much more definite and simpler. Some practical lossless networks are given to illustrate the application and validity of the proposed approach in this paper.
2. SQUARE SUBMATRIX CONSTRAINTS FOR A LOSSLESS N-PORT NETWORK

A lossless N-port network may be generally divided into two sets of ports, as shown in Fig. 1. One set has \( p \) ports, and the other set has \( q \) ports, and \( p + q = n \).

It is evident that Fig. 1 may be thought of as a generalized two-port network, whose scattering matrix can be expressed in terms of the partitioned matrix forms as follows.

\[
[S] = \begin{bmatrix}
S_{II}^{p \times p} & S_{III}^{p \times q} \\
S_{II}^{q \times p} & S_{III}^{q \times q}
\end{bmatrix}
\]  

(5)

where \( S_{II} \), \( S_{III} \), \( S_{III} \), and \( S_{III} \), are submatrixes. Here, the generalized constraints in the square submatrix case is discussed firstly, which is \( p = q \). Substituting of (5) into (1), we easily get

\[
\begin{aligned}
S_{II}^+ S_{II} + S_{III}^+ S_{III} &= I_p \\
S_{III}^+ S_{III} + S_{II}^+ S_{II} &= I_q \\
S_{II}^+ S_{II} + S_{III}^+ S_{III} &= 0
\end{aligned}
\]  

(6)

where \( I_p \) and \( I_q \) denote the \( p \times p \) and \( q \times q \) unit matrix, respectively. Let

\[
[U] = \begin{bmatrix}
S_{II} & 0 \\
0 & I_q
\end{bmatrix}
\]  

(7)

and multiply \( S \) (from the left) by \( U^+ \)

\[
[U]^+ [S] = \begin{bmatrix}
S_{II}^+ & 0 \\
0 & I_q
\end{bmatrix} \begin{bmatrix}
S_{III} & S_{III} \\
S_{III} & S_{III}
\end{bmatrix} = \begin{bmatrix}
S_{II}^+ S_{II} & S_{III}^+ S_{III} \\
S_{III} & S_{III}
\end{bmatrix}
\]  

(8)
Without changing the determinant value of the above block matrix, we may add the product of its second row with \( S_{II}^+ \) to its first row. Thus, we have

\[
\begin{align*}
\det & \begin{bmatrix} S_{II}^+ & S_{II}^+ S_{II} \\ S_{III} & S_{III} \end{bmatrix} \\
= & \begin{bmatrix} S_{II}^+ S_{II} + S_{III}^+ S_{III} & S_{II}^+ S_{II} + S_{III}^+ S_{III} \\ S_{III} & S_{III} \end{bmatrix} \\
= & \begin{bmatrix} I_p & 0 \\ S_{III} & S_{III} \end{bmatrix} = \det(S_{III})
\end{align*}
\]  

From (8), we further get

\[
\det(S_{II}^+) \det(S) = \det(S_{III})
\] (10)

Let \( \det(S_{II}) = |\det(S_{II})| e^{j\varphi_{II}} \) and \( \det(S_{III}) = |\det(S_{III})| e^{j\varphi_{III}} \), where \( \varphi_{II} \) and \( \varphi_{III} \) represent the phases of determinant of the block matrix \( S_{II} \) and \( S_{III} \), respectively. Taking \( |\det(S)| = 1 \) into account, we can get

\[
|\det(S_{II})| = |\det(S_{III})| \quad \text{(11)}
\]
\[
\det(S) = e^{j(\varphi_{II} + \varphi_{III})} \quad \text{(12)}
\]

(11) is referred to as the generalized quasi-symmetry of a lossless multiport network in partitioned matrix forms, and (12) is the generalised form of the determinant of matrix \( S \). It is worth pointing out that (11) and (12) are also true in the condition of \( p \neq q \). The third equation of (6) is rewritten as

\[
S_{II}^+ S_{III} = -S_{III}^+ S_{II}
\] (13)

Under the condition of square submatrix \( (p = q) \), it can be obtained as follows

\[
\det(S_{II}^+) \det(S_{III}) = -\det(S_{III}^+) \det(S_{III})
\] (14)

Considering (11) and (14), we easily get

\[
|\det(S_{III})| = |\det(S_{II})| \quad \text{(15)}
\]

(15) is referred to as the generalized quasi-reciprocity of a lossless multiport network in partitioned matrix forms. From (14), we can further derive the generalized characteristic phase constraint as follows

\[
\Phi_{II} = (\varphi_{III} + \varphi_{II}) - (\varphi_{II} + \varphi_{III}) = \pi
\] (16)
It is evident that the generalized characteristic phase is also equal to \( \pi \). Comparing the results of a lossless two-port network, we may get the conclusion that the magnitude and characteristic phase relations may be generalized to a lossless multiport network in square submatrix forms.

3. NON-SQUARE SUBMATRIX CONSTRAINTS FOR A LOSSLESS \( N \)-PORT NETWORK

In this section, we make a further analysis of the non-square submatrix constraints for a lossless \( N \)-port network. That is to say, an arbitrarily partitioned matrix is permitted, and \( p \neq q \). According to the unitary condition (1), we know \([S]^+ = [S]^−\). Thus, (1) can be changed into as

\[
[S][S]^+ = [I] \tag{17}
\]

which is similarly expanded in partitioned matrix forms as follows

\[
\begin{align*}
S_{II}^+ S_{II}^+ + S_{II} S_{II}^+ &= I_p \\
S_{III}^+ S_{III}^+ + S_{III} S_{III}^+ &= I_q \\
S_{III}^+ S_{II}^+ + S_{II} S_{III}^+ &= 0_{pq} \\
S_{II}^+ S_{III}^+ + S_{III} S_{II}^+ &= 0_{qp}
\end{align*}
\tag{18}
\]

From (1), we similarly get

\[
\begin{align*}
S_{II}^+ S_{II}^+ + S_{II} S_{II}^+ &= I_p \\
S_{III}^+ S_{III}^+ + S_{III} S_{III}^+ &= I_q \\
S_{III}^+ S_{II}^+ + S_{II} S_{III}^+ &= 0_{pq} \\
S_{II}^+ S_{III}^+ + S_{III} S_{II}^+ &= 0_{qp}
\end{align*}
\tag{19}
\]

where \( S_{II} \) and \( S_{III} \) are \( p \times p \) and \( q \times q \) square submatrixes, respectively. \( S_{II} \) and \( S_{III} \) are \( p \times q \) and \( q \times p \) non-square submatrixes, respectively, because \( p \neq q \). Based on (18) and (19), we can obtain the following properties.

**Property 1** Generalized quasi-reciprocity

The first equation of (18) multiplied by \( S_{II} \) from the right gives

\[S_{II} S_{II}^+ S_{II} + S_{II} S_{II}^+ S_{II} = S_{II}\]

Taking into the first equation of (19) account, we have

\[S_{II}(I_p - S_{III}^+ S_{III}) + S_{III} S_{II}^+ S_{II} = S_{II}\]
Thus, it is easily derived
\[ S_{II} S_{II}^+ S_{II} = S_{II} S_{II}^+ S_{II} \]  
(20)

Take the determinant of both sides of (20), we get
\[ \det(S_{II} S_{II}^+) = \det(S_{II} S_{II}^+) \]  
(21)

Similarly, we have
\[ \det(S_{II} S_{II}^+) = \det(S_{II} S_{II}^+) \]  
(22)

(21) and (22) are referred to as the generalized quasi-reciprocity under the condition of arbitrarily partitioned matrixes.

**Property 2** Generalized characteristic phase

From the third and fourth equations of (19), it is easily obtained
\[
\begin{align*}
S_{II}^T S_{II}^* &= -S_{II}^T S_{II}^* \\
(S_{II}^T)^* S_{II} &= -(S_{II}^T)^* S_{II}
\end{align*}
\]  
(23)

Multiply by both equations of (23), we get
\[ (S_{II}^T)^* (S_{II} S_{II}^T) S_{II}^* = (S_{II}^T)^* S_{II} S_{II}^T S_{II}^* \]  
(24)

Take the determinant of both sides of (20), we have
\[ [\det(S_{II}^*)]^2 \det(S_{II} S_{II}^T) = \det \left\{ (S_{II}^T)^* S_{II} S_{II}^T S_{II}^* \right\} \]  
(25)

Let
\[
\det(S_{II} S_{II}^T) = |\det(S_{II} S_{II}^T)| e^{j2\varphi_{II}^{OT}^{T}}
\]  
(26)

\[
\det \left\{ (S_{II}^T)^* S_{II} S_{II}^T S_{II}^* \right\} = \left| \det \left\{ (S_{II}^T)^* S_{II} S_{II}^T S_{II}^* \right\} \right| e^{j2(\varphi_{II} - \varphi_{II}^{TO}^{T})}
\]  
(27)

Substituting of (26) and (27) into (25), we can get the generalized characteristic phase constraint as follows
\[ \Phi_{II} = \left( \varphi_{II}^{TO} + \varphi_{II}^{OT} \right) - (\varphi_{II} + \varphi_{II}) = \pi \]  
(28)

Similarly,
\[ \Phi_{II} = \left( \varphi_{II}^{TO} + \varphi_{II}^{TO} \right) - (\varphi_{II} + \varphi_{II}) = \pi \]  
(29)
where $\varphi^{OT}_{III}$ and $\varphi^{TO'}_{III}$ are defined as follows

$$\det(S_{III}S_{III}^T) = |\det(S_{III}S_{III}^T)| e^{j2\varphi^{OT}_{III}}$$ (30)

$$\det\{S_{III}(S_{III}S_{III}^T)^*S_{III}\} = |\det\{S_{III}(S_{III}S_{III}^T)^*S_{III}\}| e^{j2(\varphi^{TO'}_{III} - \varphi_{III})}$$ (31)

Similarly define

$$\det(S_{III}S_{III}^*) = |\det(S_{III}S_{III}^*)| e^{j2\varphi^{TO}_{III}}$$ (32)

$$\det\{S_{III}^*(S_{III}S_{III}^*)^*S_{III}\} = |\det\{S_{III}^*(S_{III}S_{III}^*)^*S_{III}\}| e^{j2(\varphi_{III} - \varphi^{TO'}_{III})}$$ (33)

The characteristic phase constraints can be generalized into the following forms

$$\Phi_{III} = (\varphi^{OT}_{III} + \varphi^{TO'}_{III}) - (\varphi_{III} + \varphi_{III}) = \pi$$ (34)

If we define

$$\det(S_{III}^TS_{III}) = |\det(S_{III}^TS_{III})| e^{j2\varphi^{TO}_{III}}$$ (35)

$$\det\{S_{III}^*(S_{III}^TS_{III})^*S_{III}\} = |\det\{S_{III}^*(S_{III}^TS_{III})^*S_{III}\}| e^{j2(\varphi_{III} - \varphi_{III})}$$ (36)

The generalized characteristic phase relation can also be expressed as

$$\Phi_{III} = (\varphi^{TO}_{III} + \varphi^{OT'}_{III}) - (\varphi_{III} + \varphi_{III}) = \pi$$ (37)

Therefore, (28), (29), (34), and (37) are referred to as the generalized characteristic phase constraints in the arbitrarily partitioned matrix forms. It is evident that the generalized characteristic phase also equals $\pi$ in non-square submatrix forms and

$$\varphi^{TO}_{III} - \varphi^{TO'}_{III} = \varphi^{OT}_{III} - \varphi^{OT'}_{III}$$ (38)

$$\varphi^{OT}_{III} - \varphi^{OT'}_{III} = \varphi^{TO}_{III} - \varphi^{TO'}_{III}$$ (39)

### 4. LOSSLESS THREE-PORT AND FOUR-PORT NETWORK PROPERTIES

In above sections, the generalized constraint conditions for a lossless $N$-port network have been presented. Here, an analysis of the constraint conditions for three-port and four-port lossless networks is made, and the general and meaningful results are given.
Case 1 A lossless three-port network

According to the above analysis, we know \( |\det(S_{II})| = |\det(S_{III})| \).

The scattering matrix of a lossless three-port network may be partitioned into two forms

\[
[S] = \begin{bmatrix}
  s_{11} & s_{12} & s_{13} \\
  s_{21} & s_{22} & s_{23} \\
  s_{31} & s_{32} & s_{33}
\end{bmatrix}
\] (40)

\[
[S] = \begin{bmatrix}
  s_{11} & s_{12} & s_{13} \\
  s_{21} & s_{22} & s_{23} \\
  s_{31} & s_{32} & s_{33}
\end{bmatrix}
\] (41)

From the partitioned matrix of (40), it can be got the characteristic phase relation of \( \Phi_{23} \)

\[
\cos \Phi_{23} = \frac{|s_{22}|^2|s_{33}|^2 + |s_{23}|^2|s_{32}|^2 - |s_{11}|^2}{2|s_{22}||s_{33}||s_{23}||s_{32}|}
\] (42)

In addition, it can be got the characteristic phase relation of \( \Phi_{12} \) from the partitioned matrix of (41)

\[
\cos \Phi_{12} = \frac{|s_{11}|^2|s_{22}|^2 + |s_{12}|^2|s_{21}|^2 - |s_{33}|^2}{2|s_{11}||s_{22}||s_{12}||s_{21}|}
\] (43)

Since \( \det(S) = \exp\{j(\varphi_{II} + \varphi_{III})\} \), substitution of the partitioned matrix form of (40) yields

\[
\det(S) = \exp \left\{ j \left[ \varphi_{11} + \varphi_{22} + \varphi_{33} + \tan^{-1} \left( \frac{|s_{23}| |s_{32}| \sin \Phi_{23}}{|s_{22}||s_{33}| - |s_{23}||s_{32}| \cos \Phi_{23}} \right) \right] \right\}
\] (44)

Based on the submatrix form of (41), we have

\[
\det(S) = \exp \left\{ j \left[ \varphi_{11} + \varphi_{22} + \varphi_{33} + \tan^{-1} \left( \frac{-|s_{12}| |s_{21}| \sin \Phi_{12}}{|s_{11}||s_{22}| - |s_{12}||s_{21}| \cos \Phi_{12}} \right) \right] \right\}
\] (45)

Comparing (44) with (45), we easily obtain

\[
\frac{|s_{12}||s_{21}| \sin \Phi_{12}}{|s_{11}||s_{22}| - |s_{12}||s_{21}| \cos \Phi_{12}} = \frac{|s_{23}||s_{32}| \sin \Phi_{23}}{|s_{22}||s_{33}| - |s_{23}||s_{32}| \cos \Phi_{23}}
\] (46)
Case 2 A lossless four-port network

The partitioned scattering matrix of a lossless four-port network can be expressed as

$$S = \begin{bmatrix}
    s_{11} & s_{12} & s_{13} & s_{14} \\
    s_{21} & s_{22} & s_{23} & s_{24} \\
    s_{31} & s_{32} & s_{33} & s_{34} \\
    s_{41} & s_{42} & s_{43} & s_{44}
\end{bmatrix}$$  \hspace{1cm} (47)

From $|\det(S_{II})| = |\det(S_{III})|$, it is easily derived

$$|s_{11}|^2|s_{22}|^2 + |s_{12}|^2|s_{21}|^2 - 2|s_{11}||s_{22}||s_{12}|s_{21} \cos \Phi_{12}$$

$$= |s_{33}|^2|s_{44}|^2 + |s_{34}|^2|s_{43}|^2 - 2|s_{33}||s_{44}||s_{34}|s_{43} \cos \Phi_{34} \hspace{1cm} (48)$$

The characteristic phase constraint for the lossless four-port network can be expressed as

$$\Phi_{13} + \Phi_{24} + 2 \tan^{-1} \left\{ \frac{-|s_{14}||s_{23}|}{|s_{13}||s_{24}| - |s_{14}||s_{23}| \cos \left[ \frac{1}{2}(\Phi_{23} + \Phi_{14} - \Phi_{13} - \Phi_{24}) \right]} \right\}$$

$$- \tan^{-1} \left\{ \frac{-|s_{12}||s_{21}|}{|s_{11}||s_{22}| - |s_{12}||s_{21}| \cos \Phi_{12}} \right\}$$

$$- \tan^{-1} \left\{ \frac{-|s_{34}||s_{43}|}{|s_{33}||s_{44}| - |s_{34}||s_{43}| \cos \Phi_{34}} \right\} = \pi \hspace{1cm} (49)$$

The other form of the characteristic phase constraint can be written as follows

$$\Phi_{23} + \Phi_{14} + 2 \tan^{-1} \left\{ \frac{-|s_{13}||s_{24}|}{|s_{14}||s_{23}| - |s_{13}||s_{24}| \cos \left[ \frac{1}{2}(\Phi_{13} + \Phi_{24} - \Phi_{23} - \Phi_{14}) \right]} \right\}$$

$$- \tan^{-1} \left\{ \frac{-|s_{12}||s_{21}|}{|s_{11}||s_{22}| - |s_{12}||s_{21}| \cos \Phi_{12}} \right\}$$

$$- \tan^{-1} \left\{ \frac{-|s_{34}||s_{43}|}{|s_{33}||s_{44}| - |s_{34}||s_{43}| \cos \Phi_{34}} \right\} = \pi \hspace{1cm} (50)$$

5. NUMERICAL EXAMPLES

In this paper, the generalized constraints for a waveguide T-junction with a septum are analyzed, which is a lossless three-port even symmetry network shown in Fig. 2(a).
According to the properties of even symmetry, the scattering matrix can be expressed as

\[
[S] = \begin{bmatrix}
|s_{11}|^2 & s_{12} & s_{12} \\
 s_{12} & |s_{22}|^2 & s_{23} \\
 s_{12} & s_{23} & |s_{22}|^2
\end{bmatrix}
\]

From (42) and (43), we easily get

\[
\cos \Phi_{12} = \frac{|s_{11}|^2|s_{22}|^2 + |s_{12}|^2 - |s_{22}|^2}{2|s_{11}| |s_{22}||s_{12}|^2} \quad (51)
\]

\[
\cos \Phi_{23} = \frac{|s_{22}|^4 + |s_{23}|^4 - |s_{11}|^2}{2|s_{22}|^2|s_{23}|^2} \quad (52)
\]

We give a numerical analysis of the $T$-junction with a septum in the
Generalized constraint for a lossless multiport network

X-band rectangular waveguide (WR-90) by using the planar circuit method [8–10]. The sizes of the T-junction are $L = 80$ mm, $t = 1.0$ mm, $H = 60$ mm, $l = 11.43$ mm, respectively, as shown in Fig. 2(b). The network $S$ parameters at the center operational frequency, $f_0 = 9.375$ GHz, are calculated as follows [9]

$$
\begin{align*}
|s_{11}| &= 0.1066444 \quad \varphi_{11} = -18.30309^\circ \\
|s_{22}| &= 0.5138444 \quad \varphi_{22} = -109.18662^\circ \\
|s_{12}| &= 0.7031113 \quad \varphi_{12} = 62.28501^\circ \\
|s_{23}| &= 0.4915795 \quad \varphi_{23} = 58.90225^\circ
\end{align*}
$$

(53)

$\Phi^0_{ij}$ is used to indicate the characteristic phase calculated with the phases of (53) directly, and $\Phi_{ij}$ indicate the characteristic phase obtained with the magnitudes of $S$ parameters using the (51) and (52). The comparison of $\Phi^0_{ij}$ with $\Phi_{ij}$ is given in Table 1. We assume the variable range of the characteristic phases is from 0 to 360 degrees.

<table>
<thead>
<tr>
<th>$\Phi_{12}$</th>
<th>$\Phi_{13}$</th>
</tr>
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<tbody>
<tr>
<td>$= 2\varphi_{12} - (\varphi_{11} + \varphi_{22})$</td>
<td>$\cos^{-1}\left(\frac{</td>
</tr>
<tr>
<td>$\Phi_{12}$</td>
<td>252.05973$^\circ$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\Phi_{23}$</th>
<th>$\Phi_{23}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$= 2(\varphi_{23} - \varphi_{22})$</td>
<td>$\cos^{-1}\left(\frac{</td>
</tr>
<tr>
<td>$\Phi_{23}$</td>
<td>336.17774$^\circ$</td>
</tr>
</tbody>
</table>

By comparison $\Phi^0_{ij}$ with $\Phi_{ij}$, it can be found that they are in great agreement. Therefore, we may believe that the numerical results calculated by using the planar circuit method are correct. But it can be also estimated that the magnitude of $s_{12}$ has a little error, which results in the error of the characteristic phase $\Phi_{12}$. In addition, according to

$$(S_{III}^T S_{IIII}^*) (S_{IIII}^T S_{III}) = S_{II}^T (S_{III} S_{IIII}^T)^* S_{II1},$$

we have

$$s_{12}^2 (s_{22}^* + s_{23}^*)^2 = s_{12}^* s_{11}^2$$

(54)
The generalized characteristic phase relation can be expressed as

\[ 2\varphi_{12} - (\varphi_{11} + \varphi_{22}) + \tan^{-1}\left(\frac{-|s_{23}| \sin\left(\frac{1}{2}\Phi_{23}\right)}{|s_{22}| + |s_{23}| \cos\left(\frac{1}{2}\Phi_{23}\right)}\right) = \pi \] (55)

Substituting (53) into (55), the numerical result shows that equation (55) is true.

On the other hand, it is well known that all instruments available have the measurement errors of magnitude and phase. Therefore, a reasonable estimation of the experimental errors needs to be made in some cases. The generalized constraint for characteristic phases can help us to give a reasonable estimation of the experimental errors. Here, the error estimation of the characteristic phases is illustrated with \( \Phi_{23} \), which is one of the characteristic phases of the three-port even symmetry network. For simplicity, we let

\[ \Theta_{23} = \frac{1}{2}\Phi_{23} = \cos^{-1}\left(\frac{|s_{12}|^2}{2|s_{11}||s_{22}|}\right) \] (56)

and

\[ \Delta\Theta_{23} = \Theta_{23}^0 - \Theta_{23} \] (57)

where \( \Theta_{23}^0 = \varphi_{23} - \varphi_{22} \) and \( \Delta\Theta_{23} \) indicates the error of the characteristic phases, which results from the error \( \Delta\Theta_{23}^0 \) of phases itself and the error \( \Delta\Theta_{23}^A \) made by magnitudes, generally, \( \Delta\Theta_{23} \geq 0 \). If we assume the phase error \( \Delta\varphi \) of each of \( S \) parameters is identical, and the relative errors of magnitudes \( \Delta A \) of each of \( S \) parameters are also identical, it can be easily derived

\[ \frac{1}{2}\Delta\Theta - 2\frac{\Delta A}{A} \frac{|s_{12}|^2}{2|s_{11}||s_{22}|} \left(1 - \left(\frac{|s_{12}|^2}{2|s_{11}||s_{22}|}\right)^2\right)^{\frac{1}{2}} < \Delta\varphi \]

\[ < \frac{1}{2}\Delta\Theta + 2\frac{\Delta A}{A} \frac{|s_{12}|^2}{2|s_{11}||s_{22}|} \left(1 - \left(\frac{|s_{12}|^2}{2|s_{11}||s_{22}|}\right)^2\right)^{\frac{1}{2}} \] (58)

According to (58), we may make an estimation of the phase errors of \( S \) parameters to some extent from the experiment results and the magnitude errors gauged by the instruments.
6. CONCLUSION

An arbitrary lossless multiport network can be regarded as a generalized two-port network by introducing the partitioned matrix description. This paper has theoretically proven that the constraint conditions for a lossless two-port network can be generalized to a lossless multiport network, and thus the generalized quasi-reciprocity, quasi-symmetry, and the characteristic phase relation are derived efficiently by using the partitioned matrix forms, which may be used to analyze the particular properties of the some lossless multiport networks more effectively. In addition, the relation of characteristic phases can be used to check up the equivalent network $S$ parameters obtained by the numerical calculations or the experiment measurement to some extent.

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